On the stability of fully developed flow in a pipe

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The stability of infinitesimal axially symmetric disturbances in fully developed pipe flow is examined anew. The classical eigenvalue problem is treated in part by asymptotic methods and leads to an algebraic relation between the eigenvalue c, the disturbance wavelength $2\pi/\alpha$, and the Reynolds number. Examination of the limiting cases of this relation reveals the existence of two families of characteristic numbers, the value of which tends to unity and to zero as the Reynolds number increases without bounds. For the latter, a more accurate solution is required and given. It is found that all eigenvalues yield stable solutions and that for a given wave number and Reynolds number only a finite number of eigenvalues exists.

The limitations of the analysis are discussed in the light of a recent experimental study of the same problem.

1. Introduction

It is generally believed that fully developed flow in a pipe is stable at any Reynolds number when it is excited by infinitesimally small axially symmetric disturbances. While several authors have considered this problem, the proof of stability offered by Sexl (1927, 1928) has served as a standard reference. A recent investigation by Pekeris (1948*a*) points out that certain *a priori* assumptions about the nature of the stability criteria (the eigenvalues) damaged the rigour of Sexl's proof. The same paper gave an independent demonstration of stability. It has been shown that Sexl's work may be criticized on other grounds as well (Corcos 1952), and the present paper shows that Pekeris' work is incomplete in the sense that he only investigated one class or family of modes while two distinct classes of modes can be found. These two sets of modes are very similar to those found by Pekeris (1948*b*) in connexion with two-dimensional Poiseuille flows. In the latter problem, one of the sets is always stable while the other contributes the unstable perturbations at some Reynolds numbers. In our case, as the following development will undertake to show, both sets are stable or positively damped.

Nevertheless, the second set of modes deserves to be studied because it generates the least stable disturbances, because it is germane to the unstable set of the two-dimensional problem, and because it corresponds rather well with a set of modes which is observed experimentally (Leite 1959).

In the following, the eigenvalues corresponding to any mode are given as functions of Reynolds number and wavelength, and it is shown that the number of modes present (i.e. the number of eigenvalues) is finite and depends for a given wavelength of the disturbances upon the Reynolds number.

2. The formulation of the problem

Let the mean flow be an axially symmetric pipe flow, and let the disturbances be axially symmetric and periodic in the direction of flow, i.e. the stream function has a perturbation of the form

$$\psi(x,r,t) = \phi(r)\, e^{i\alpha(x-ct)}.$$

Then the appropriate differential equation for the eigen function $\phi(r)$ (Sexl 1927) is

$$(W-c)\left(\phi''-\frac{\phi'}{r}-\alpha^2\phi\right)+\phi\left(\frac{W'}{r}-W''\right)$$
$$= -\frac{i}{\alpha R}\left[\phi^{\mathrm{iv}}-\frac{2}{r}\phi'''+\frac{3}{r^2}\phi''-\frac{3}{r^3}\phi'-2\alpha^2\left\{\phi''-\frac{\phi'}{r}\right\}+\alpha^4\phi\right]. \quad (2.1)$$

Here $c = c_r+ic_i$ is the eigenvalue, α is the wave-number of the disturbance, $W = W(r)$ is the non-dimensional main stream velocity profile, and R = (centre velocity) (radius of pipe)/(kinematic viscosity) is the Reynolds number of the flow. Primes indicate differentiation with respect to r. The boundary conditions at the origin (centre of the pipe) require that the disturbance be axially symmetric and bounded, i.e.

$$\lim_{r\to 0}\frac{\phi}{r} = 0 \quad \text{and} \quad \lim_{r\to 0}\frac{\phi'}{r} \quad \text{is bounded.} \quad (2.2a)$$

On the walls, the kinematic and no-slip conditions are equivalent to

$$\phi(1) = \phi'(1) = 0. \quad (2.2b)$$

A stable disturbance is characterized by $c_i < 0$, a neutral disturbance by $c_i = 0$, and an unstable disturbance by $c_i > 0$.

In general, the differential equation (2.1) will have four linearly independent solutions. The method of approach adopted herein will enable us to eliminate immediately those solutions which are not sufficiently regular near the origin. The remaining two solutions we will denote by ϕ_i and ϕ_a. Then the appropriate solution may be written as a linear combination of ϕ_i and ϕ_a

$$\phi = C_1\phi_i + C_2\phi_a. \quad (2.3)$$

Applying the boundary conditions (2.2), we will obtain two homogeneous equations for the constants C_1 and C_2. In order for a non-trivial solution to exist, the determinant of the coefficients C_1 and C_2 must vanish. We thus obtain

$$F(\alpha,\alpha R,c) = \begin{vmatrix} \phi_i(1) & \phi_a(1) \\ \phi_i'(1) & \phi_a'(1) \end{vmatrix} = 0. \quad (2.4)$$

This equation gives the functional dependence of c on α and αR and thus indicates for what Reynolds number, if any, disturbances will grow in time.

3. The determination of the 'inviscid' solutions

For Poiseuille flow, $W(r) = 1-r^2$, and the differential equation (2.1) may be written conveniently as

$$f'' - \frac{1}{r}f' - \alpha^2 f - i\alpha R(1-r^2-c)f = 0, \tag{3.1}$$

where

$$\phi'' - \frac{\phi'}{r} - \alpha^2\phi = f. \tag{3.2}$$

This form has been previously utilized by both Sexl and Pekeris.

Clearly $f = 0$ is a solution of (3.1); substituting in (3.2) we obtain ϕ in terms of Bessel functions of order unity, of the first and of the second kind, i.e.

$$\phi_1 = rJ_1(i\alpha r), \quad \phi_2 = rY_1(i\alpha r). \tag{3.3}$$

These are two linearly independent exact solutions of (2.1), and they are usually referred to as the inviscid solutions, since they are solutions of the perturbation equation for $R = \infty$. The second solution ϕ_2 is rejected immediately in conformity with the boundary conditions at the origin, and thus we obtain

$$\phi_i = rJ_1(i\alpha r). \tag{3.4}$$

4. The determination of the second independent solution

Equation (2.1) has power series solutions which are convergent throughout the interval $0 \leqslant r \leqslant 1$. However, these solutions converge very slowly because positive powers of αR (a large number) occur as coefficients in the series. This type of expansion is therefore of little use to us for an explicit solution. Instead, we resort to an asymptotic development due to Heisenberg. We look for a solution of the form

$$\phi = e^g, \tag{4.1}$$

and we assume that for large enough values of αR we can express g by the asymptotic series

$$g = (\alpha R)^n g_0 + g_1 + (\alpha R)^{-n} g_2 + \dots. \tag{4.2}$$

Substituting (4.1) and (4.2) in (2.1) and equating coefficients of equal powers of αR, we find that

$$n = \tfrac{1}{2}; \quad g_0 = \pm\int_{r_0}^{r} \sqrt{\{i(W-c)\}}\,dr; \quad g_1 = \log\frac{r^{\frac{1}{2}}}{(W-c)^{\frac{5}{4}}}.$$

Making use of the first and second approximations only, we write

$$\phi_a = r^{\frac{1}{2}}(W-c)^{-\frac{5}{4}}[A\chi_+(r_0, r) + B\chi_-(r_0, r)], \tag{4.3}$$

where

$$\chi_+(a,b) = \exp\left\{+\int_a^b \sqrt{\{i\alpha R(W-c)\}}\,dr\right\},$$

$$\chi_-(a,b) = \exp\left\{-\int_a^b \sqrt{\{i\alpha R(W-c)\}}\,dr\right\},$$

and where the ratio A/B will now be determined around the origin by comparing ϕ_a with a regular (viscous) solution valid in the neighbourhood of $r = 0$.

We note that $i\alpha R(1-r^2-c) \doteqdot i\alpha R(1-c)$ for small values of r. Making this approximation in (3.1), and neglecting α^2 compared to αR, we can write the solution of (3.1) directly in terms of Bessel functions of order one. Thus

$$f = hrJ_1(i\gamma r) + lrY_1(i\gamma r), \tag{4.4}$$

where $\gamma = \{i\alpha R(1-c)\}^{\frac{1}{2}}$ and h and l are constants. From (3.2), this gives directly

$$\phi_a = h_1 rJ_1(i\gamma r) + l_1 rY(i\gamma r) \tag{4.5}$$

provided $\gamma \neq \alpha$, which will always be true in any of the cases of interest.

The boundary conditions at the origin require that l_1 be zero, and thus ϕ_a must be represented near the origin by

$$\phi_a = h_1 rJ_1(i\gamma r). \tag{4.6}$$

Even though this representation is restricted to small values of r, if αR is sufficiently large, we may make γr large enough to use the asymptotic form of the above Bessel function. Then we have

$$\phi_a = h_2 r^{\frac{1}{2}}[\exp\{-\langle i\alpha R(1-c)\rangle^{\frac{1}{2}} r + \tfrac{3}{4}\pi i\} + \exp\{+\langle i\alpha R(1-c)\rangle^{\frac{1}{2}} - \tfrac{3}{4}\pi i\}], \tag{4.7}$$

provided $$-\pi < \arg i\{i\alpha R(1-c)\}^{\frac{1}{2}} < 2\pi.$$

On the other hand, if $r_0 = 0$, (3.3) becomes for very small r (bounded from zero)

$$\phi_a = r^{\frac{1}{2}}(1-c)^{-\frac{5}{4}}[A_0 \exp\{+\langle i\alpha R(1-c)\rangle^{\frac{1}{2}} r\} + B\exp\{-\langle i\alpha R(1-c)\rangle^{\frac{1}{2}} r\}], \tag{4.8}$$

where the notation A_0 and B_0 is intended to indicate that r_0 in (4.3) is taken as zero. A comparison of (4.8) and (4.7) gives

$$\frac{A_0}{B_0} = i. \tag{4.9}$$

The ratio A/B has now been evaluated so that ϕ_a corresponds to the solution with the proper behaviour near the origin, but we must impose certain restrictions upon the use of (4.3) with (4.9). First, it is clear that (4.3), an asymptotic approximation for large αR, fails at $r = 0$; although it is valid (in the sense of being a good approximation) arbitrarily close to $r = 0$ for sufficiently large αR. This point will be dealt with later. Then since (4.3) contains a singularity for $W = c$, which is a regular point of the differential equation (2.1), the asymptotic representation fails there also. Finally, we should not expect the ratio A/B as found in (4.9) to remain the same for the whole interval, particularly in the neighbourhood of the singularity $W = c$. The reason can be set forth as follows. Our asymptotic representation must be single-valued if it is to approximate an exact solution of (2.1). On the other hand, the functions χ_+ and χ_- have branch cuts and are multi-valued functions of the complex argument r.

Thus a single linear combination of χ_+ and χ_- would yield a non-unique solution. On the other hand, it is possible to chose the coefficients of χ_+ and χ_- corresponding to various parts of the complex r plane in such a manner as to satisfy the two requirements that we must place upon the solution: (*a*) that it be unique; (*b*) that it be continuous, or at least that discontinuities in the solution be of the order of the errors made by an asymptotic approximation.

The question is examined in detail in the Appendix. The need to modify the linear combinations of asymptotic solutions here and there (in our case on three discrete lines) was first described by Stokes, and the lines on which solutions are connected to each other are termed the Stokes lines.

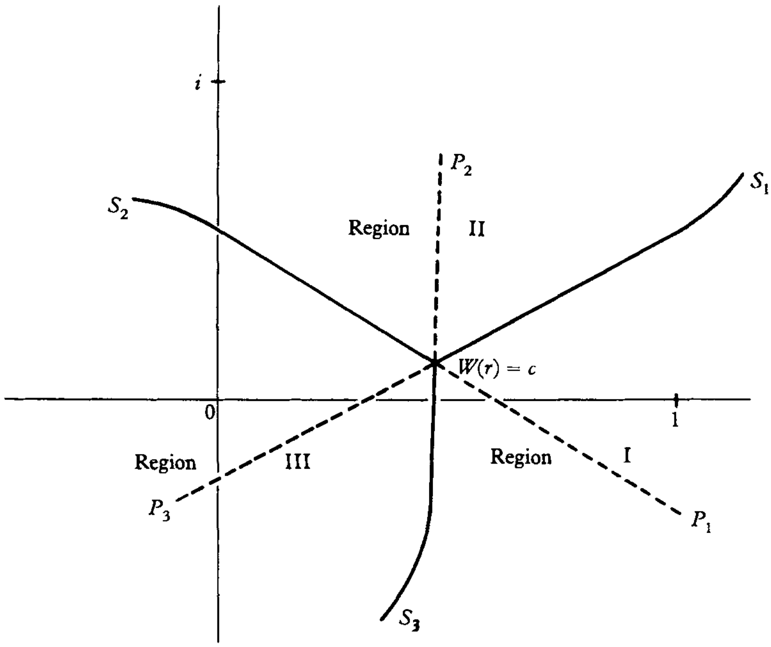

FIGURE 1. The complex r-plane and the Stokes regions.

The Appendix shows that Stokes lines originate at the complex value of r for which $W = c$, that they divide the complex r plane into three regions (figure 1), and that we should expect $r = 0$ to be in Region III and $r = 1$ to be in Region I. Furthermore, the connexion formulae taken together with (4.9) yield

$$\frac{A_{\mathrm{I}}}{B_{\mathrm{I}}} = i[1-e^{Q}], \tag{4.10}$$

where
$$Q = 2\int_k^0 \sqrt{\{i\alpha R(W-c)\}}\,dr = e^{\frac{5}{4}\pi i}(\alpha R)^{\frac{1}{2}}\,\tfrac{1}{2}\pi k^2, \tag{4.11}$$

with
$$k^2 = 1-c.$$

5. The eigenvalue equation

Henceforth in evaluating integrals of the type

$$\int_k^1 \sqrt{\{i\alpha R(W-c)\}}\,dr \quad \text{and} \quad \int_k^0 \sqrt{\{i\alpha R(W-c)\}}\,dr,$$

we shall choose our arguments in conformity with the convention adopted when we defined the Stokes regions (see Appendix). Proceeding with this in mind, we may write the determinant (2.4) explicitly. We thus obtain an algebraic eigenvalue equation. Simplifying a little, we get

$$(1-e^{Q})\,e^{P} = -i\left[\frac{e^{-\frac{1}{4}\pi i}\sqrt{(\alpha Rc)}+\dfrac{[d/dr\{rJ_1(i\alpha r)\}]_{r=1}}{J_1(i\alpha)}+\dfrac{5}{2c}-\dfrac{1}{2}}{e^{-\frac{1}{4}\pi i}\sqrt{(\alpha Rc)}-\dfrac{d/dr\{rJ_1(i\alpha r)\}_{r=1}}{J_1(i\alpha)}-\dfrac{5}{2c}+\dfrac{1}{2}}\right], \tag{5.1}$$

where $$P = \int_k^1 \sqrt{\{i\alpha R(W-c)\}}\,dr = e^{-\frac{1}{4}\pi i}\sqrt{(\alpha R)}\left[c^{\frac{1}{2}}-k^2\log\frac{1+c^{\frac{1}{2}}}{k}\right],$$

and where we have taken

$$-\pi < \arg(c) \leqslant \pi \quad \text{and} \quad -\pi < \arg k \leqslant \pi.$$

The solution of (5.1) will determine c as a function of α and of αR.

Equation (5.1) has to be solved by trial and error for any value of α and αR, and a representative tabulation would be very time consuming. However, the important conclusions can be drawn without resorting to extensive numerical work. It is sufficient to note the general behaviour of the eigenvalues and to develop formulae for the interesting limiting cases.

We shall successively assume that c is non-vanishing as $\alpha R \to \infty$ and that $c \to 0$ as $\alpha R \to \infty$, and we shall thus get two different limiting representations for the eigenvalues. These representations we shall call families of eigenvalues.

6. The first limiting form

Let us first assume that c is non-vanishing as $\alpha R \to \infty$. Then for large αR the right-hand side of (5.3) is approximately $-i$, while the term $\exp(P)$ grows very large. Thus the term $(1-e^{Q})$ must be very nearly zero. We have for this case

$$e^{Q} = 1,$$
$$Q = -2\pi N i,$$
$$c = 1+\frac{4N}{(\alpha R)^{\frac{1}{2}}}e^{-\frac{1}{4}(3\pi i)} \quad (N = 1,2,3\ldots). \tag{6.1}$$

It is seen that all values of c given by (6.1) correspond to damped disturbances. We have taken only positive values of N in (6.1) because negative values would give values of c which are inconsistent with our connexion formulae.

The family of eigenvalues given by (6.1) requires that $c \to 1$ as $\alpha R \to \infty$. As a consequence, the Stokes point approaches the boundary $r = 0$ for increasing values of αR and the following question should be raised. Is the matching of our asymptotic solutions with Bessel functions in the neighbourhood of the origin legitimate? In the neighbourhood of the origin the argument of the Bessel functions used is $i\{i\alpha R(1-c)\}^{\frac{1}{2}}r$. The question is whether there is in Region III a neighbourhood of $r = 0$ for which the argument is large enough to justify the use of asymptotic expansions. Substituting for c from (6.1), we get for the representation in the neighbourhood of the origin

$$\phi_a = rJ_1[E(\alpha R)^{\frac{1}{4}}r],$$

where E is a constant. Now it is true that $r = 0$ is, in addition, only a distance of order $(\alpha R)^{-\frac{1}{3}}$ away from the Stokes point. However, use is made of the asymptotic expansions not at $r = 0$ but in a neighbourhood of it which can be chosen for convenience anywhere in Region III provided that in this neighbourhood $r \ll 1$, say, $|r| \gg (\alpha R)^{-\frac{1}{3}}$. It is therefore evident that the matching to the Bessel function at the origin can always be made even for the family of eigenvalues (6.1), provided αR is large enough. In addition, an obvious restriction to the use of (6.1) is that $4N \ll \alpha R$, so that only a relatively small number of modes is given with accuracy by (6.1).

Pekeris (1948*a*) derived an equation identical to (6.1) to the proper order of approximation by an independent method.* However, this method did not yield the other family of eigenvalues, namely, that for which $c \to 0$ as $\alpha R \to \infty$. This is the case to which we now turn.

7. The second limiting form

For small c we find that

$$P \simeq \tfrac{2}{3}(\alpha R)^{\frac{1}{2}} c^{\frac{3}{2}} e^{-\frac{1}{4}i\pi},$$

and note that

$$e^{Q} \to 0 \quad \text{as} \quad \alpha R \to \infty.$$

Defining

$$\delta = (2i\alpha R)^{\frac{1}{3}} c,$$

we are able to write the eigenvalue equation in the following form:

$$\exp\left\{\frac{\sqrt{2}}{3} i\delta^{\frac{3}{2}}\right\} = -i\left\{\frac{i2^{\frac{1}{2}}\delta^{\frac{3}{2}} - 5}{i2^{\frac{1}{2}}\delta^{\frac{3}{2}} + 5}\right\}.$$

This leads to the formula

$$\delta^{\frac{3}{2}} = \frac{3\pi}{2^{\frac{3}{2}}}\left[4N + 1 - \frac{4\theta}{\pi}\right] \quad (N = 0, 1, 2 \ldots), \tag{7.1}$$

where

$$\theta = \arctan \frac{\sqrt{2}}{5}\delta^{\frac{3}{2}}.$$

There are discrete values of δ corresponding to the various modes; and as δ increases, $4\theta/\pi$ rapidly tends to 2, so that for the higher modes

$$c = \frac{1}{2(2i\alpha R)^{\frac{1}{3}}}[3\pi(4N+1)]^{\frac{2}{3}}. \tag{7.2}$$

Again, we see that we have a set of eigenvalues corresponding to damped disturbances. We must again restrict the value of N in (7.1) so that $|c| \ll 1$. Figure 2 now gives an approximate picture of the eigenvalues in the complex c-plane for a given value of αR. At each end of the interval there are a finite number of eigenvalues described by (6.1) and (7.1). In between we cannot simplify the eigenvalue equation, but a few numerical computations at given Reynolds numbers suffice to give the behaviour of the solution. The results indicate that in the middle region there is again a finite number of eigenvalues. The actual number

* Pekeris' third term (1948*a*, equation (44)) is of the order of $(\alpha^2/\alpha R)$ and can be obtained by the present method if one proceeds as above after defining in (2.1) and (3.1) a quantity $c = c + i\alpha^2/\alpha R$, instead of neglecting α^2 compared to $i\alpha R$.

of solutions possible for a given Reynolds number and a given wave-number is relatively small but increases, of course, as αR increases. For instance, there are about eighteen eigenvalues for $\alpha R = 10{,}000$.

The same sort of objection can be made to (7.1) as was made to (6.1). The Stokes point approaches the boundary $r = 1$ as $R \to \infty$; hence, is the asymptotic representation valid? Here the objection is more serious, and a little reasoning (Corcos 1952, p. 25) shows that the argument of the asymptotic expressions is, at least for lower modes (N small), of the order of unity no matter how large αR is. Therefore, we will now obtain a more accurate representation of the family for which $c \to 0$.

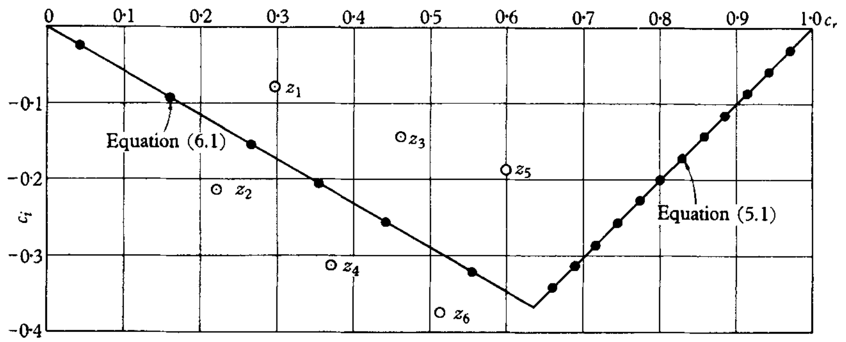

FIGURE 2. c_i *vs* c_r, for $\alpha R = 10{,}000$.

8. Expansion in terms of a convergent series

We will attempt to connect the preceding work with an alternate development which should give a more accurate representation of the viscous solution when the boundary is in the immediate vicinity of the Stokes point. We will use a notation suggestive of the corresponding quantities in the work of Lin (1945) in order that some of his results may be utilized.

We will first translate our origin of co-ordinates to the wall so that the slope of the velocity profile will be positive near the wall to correspond to the convention used in plane Poiseuille flow. We define

$$y = 1 - r.$$

Thus, the Stokes point is given by

$$y_k = 1 - k.$$

Further, following Lin, we take

$$\epsilon\eta = (y - y_k)$$

and

$$(W - c) = W'_k \eta\epsilon + \frac{W''_k}{2!}(\eta\epsilon)^2 + \ldots,$$

where $\epsilon = (\alpha R)^{-\frac{1}{3}}$. We then express (2.1) in terms of η and look for a solution of the form

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots.$$

We find that ϕ_0 satisfies the differential equation

$$-i\phi_0^{\text{iv}} = W'_k \eta \phi''.$$

The viscous solutions are given by the particular solutions (Lin 1945; equation (4.9))

$$\phi_3 = \int_{+\infty}^{\eta} d\xi \int_{+\infty}^{\xi} d\mu\, \mu^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}[\tfrac{2}{3}(i\alpha_0 \mu)^{\frac{3}{2}}]$$

and
$$\phi_4 = \int_{-\infty}^{\eta} d\xi \int_{-\infty}^{\xi} d\mu\, \mu^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)}[\tfrac{2}{3}(i\alpha_0 \mu)^{\frac{3}{2}}], \tag{8.1}$$

where
$$\alpha_0 = \left(\frac{dW}{dy}\right)^{\frac{1}{3}}_{y=y_k},$$

and $H_{\frac{1}{3}}^{(1,2)}$ represent Hankel functions of the first and second kind of order one-third. Our problem is to connect these solutions with the asymptotic viscous solutions. For small values of c we have seen that e^Q is very small. If e^Q is taken as being exactly zero, we have in Region I (near the wall)

$$\phi_a = A_1 \frac{r^{\frac{1}{2}}}{(W-c)^{\frac{5}{4}}} [\chi_+(k,r) - i\chi_-(k,r)]. \tag{8.2}$$

We may ask what combination corresponds to this in Region III. From (8.2) we take $B_1 = iA_{\mathrm{I}}$ in the connexion formula (4.11), which yields

$$A_{\mathrm{III}} = A_{\mathrm{I}}, \quad B_{\mathrm{III}} = 0.$$

This shows that near the Stokes point only one of the viscous terms in Region III has significance. We then have

$$\phi_a = \frac{Ar^{\frac{1}{2}}}{(W-c)^{\frac{5}{4}}} \chi_+(k,r)$$

for r near the Stokes point in Region III. Near the Stokes point this solution may be rewritten approximately as

$$\phi_a = \text{const.} \times \frac{e^{\frac{2}{3}i(i\alpha_0\eta)^{\frac{3}{2}}}}{\eta^{\frac{5}{4}}}.$$

Similarly, ϕ_3 as given by (8.1) may be expressed asymptotically for large values of η by

$$\phi_3 = \text{const.} \times \frac{e^{\frac{2}{3}i(i\alpha_0\eta)^{\frac{3}{2}}}}{\eta^{\frac{5}{4}}}.$$

So it can be seen that the viscous solution is expressed (to the proper order of approximation) by ϕ_3. Since ϕ_3 is analytic in the region about the Stokes point it must represent the solution everywhere in that neighbourhood. In particular, it is valid on the boundary, and we may formulate our eigenvalue determinant in terms of this solution and the inviscid solution $rJ_1(i\alpha r)$.

9. An alternative determination of the eigenvalues

Writing the eigenvalue problem in terms of ϕ_3 leads us to the expression

$$\left(\frac{\phi_3}{\phi_3'}\right)_{y=0} = (k-1)\,F(z) = \frac{-J_1(i\alpha)}{[d/dr\{rJ_1(i\alpha r)\}]_{r=1}}, \tag{9.1}$$

where $z = (1-k)(\alpha R)^{\frac{1}{3}}(2k)^{\frac{1}{3}}$ and $F(z)$ is the Tietjens function

$$F(z) = \frac{\int_{\infty}^{-z} d\xi \int_{\infty}^{\xi} d\mu\, \mu^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}[\frac{2}{3}(i\mu)^{\frac{3}{2}}]}{-z\int_{\infty}^{-z} d\mu\, \mu^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}[\frac{2}{3}(i\mu)^{\frac{3}{2}}]}$$

which has been tabulated by Tietjens and by Lin for real values of z corresponding to neutral disturbances. Since

$$J_1(i\alpha)/[d/dr\{rJ(i\alpha r)\}]_{r=1}$$

is real and positive for all values of α, a neutral disturbance would require that $F(z)$ be real and positive for z positive. According to the tabulation of $F(z)$ given by Lin (1945, p. 140), this can only occur at $z = 2{\cdot}5$, $F(z) = 0{\cdot}51$. These values substituted in (9.1) are found to require α to be of the order of $(\alpha R)^{\frac{1}{3}}$. For this case the whole treatment is invalid. However, the work of Synge (1939) indicates that very short wavelength disturbances can be expected to be stable from energy considerations. Thus, we conclude that there is no neutral disturbance. A very interesting and significant comparison can be made between this family of eigenvalues and those of Couette flow. The inviscid solutions for Couette flow are $\sinh\alpha y$ and $\cosh\alpha y$. Using these, Lin's formula (1945, equation (6.13)) becomes

$$\left(\frac{\phi_3}{\phi_3'}\right)_{y=0} = -\frac{\sinh\alpha}{\alpha\cosh\alpha} - \left(\frac{\phi_4}{\phi_4'}\right)_{y=1}\frac{\cosh\alpha}{\alpha\sinh\alpha}. \tag{9.2}$$

If we look for solutions of this equation for which c is small, the second right-hand term can be neglected since

$$\left(\frac{\phi_4}{\phi_4'}\right)_{y=1} = \theta[(\alpha R)^{-\frac{1}{3}}].$$

The resulting eigenvalue problem

$$\frac{\phi_3(0)}{\phi_3'(0)} = -\frac{\sinh\alpha}{[d/dy(\sinh\alpha y)]_{y=1}} \tag{9.3}$$

is remarkably similar to equation (9.1). This means that Couette flow has one set of modes given by an equation similar to equation (7.2). Another set of modes will be found for which $c \to 1$ as $\alpha R \to \infty$. These two sets correspond to the dominance of the term

$$\frac{\phi_3(0)}{\phi_3'(0)},$$

or the term

$$\frac{\phi_4(1)}{\phi_4'(1)}$$

respectively in equation (9.2). For c in the middle region, these terms are of equal importance and the whole equation must be considered. The eigenvalues are given qualitatively by a figure similar to figure 2, but are symmetrical about $c = \frac{1}{2}$. We may now determine the eigenvalues of both Couette flow and pipe flow for which $c \to 0$ as $\alpha R \to \infty$, using the Tietjens function. Since the terms involving the wave number are independent of αR and c, $F(z)$ must grow large without bounds if $y-y_k \to 0$. This happens when z satisfies the equation

$$\int_{\infty}^{-z} d\mu \, \mu^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}[\tfrac{2}{3}(i\mu)^{\frac{3}{2}}] = 0, \tag{9.4}$$

where for Poiseuille flows

$$z = [1-\sqrt{(1-c)}]\,(1-c)^{\frac{1}{6}}\,2^{\frac{1}{3}}(\alpha R)^{\frac{1}{3}},$$

and for Couette flows

$$z = c(\alpha R)^{\frac{1}{3}}.$$

Equation (9.4) gives a discrete set of values for z, which we will denote by z_n. These zeros lie in pairs which are very nearly reflexions in the ray $(\arg z) = \frac{1}{6}\pi$. The first pair is approximately

$$z_{1,2} = 4{\cdot}23e^{(-\frac{1}{6}\pi \pm \frac{1}{12}\pi)i}, \tag{9.5}$$

a result obtained by a numerical integration using tabulated values of the integrand (Furry 1945). Other values are

$$z_{3,4} = 6{\cdot}8e^{(-\frac{1}{6}\pi \pm 12^\circ)i}, \quad z_{5,6} = 9{\cdot}5e^{(-\frac{1}{6}\pi \pm 8^\circ)i}.$$

The most critical of these values, $z_i = 4{\cdot}23e^{-\frac{1}{12}\pi i}$, was checked on a differential analyser; the others are only rough estimates. Figure 2 is a plot of the eigenvalues corresponding to $\alpha R = 10{,}000$ and belonging to both families (i.e. the family $c \to 0$ as $\alpha R \to \infty$ obtained by integrating (9.4) and the family $c \to 1$ as $\alpha R \to \infty$ given by (6.1)). In the region where the two families join, neither representation is accurate, and the complete equation (5.1) should be used. Nevertheless, figure 2 illustrates quite well the fact that there exists only a finite number of modes for any αR.

It is interesting to note that for the first few modes an asymptotic formula such as the one given by Hopf (1914) in the case of Couette flow is very inaccurate. For $n > 6$ the roots of (9.4) are given to sufficient accuracy by the asymptotic equation

$$\exp\{\tfrac{2}{3}e^{-\frac{1}{4}\pi} z^{\frac{3}{2}} - \tfrac{5}{12}\pi i\} - \exp\{-\tfrac{2}{3}e^{-\frac{1}{4}i\pi} z^{\frac{3}{2}} - \tfrac{11}{12}\pi i\} + \frac{3{\cdot}03}{0{\cdot}853} z^{\frac{5}{4}} e^{\frac{17}{24}\pi i} = 0,$$
$$\tfrac{1}{2}\pi > (\arg z) > -\tfrac{2}{3}\pi. \tag{9.6}$$

Thus

$$z_1 = 4{\cdot}23e^{-\frac{1}{12}\pi i} = \{1-\sqrt{(1-c)}\}\,(1-c)^{\frac{1}{6}} z^{\frac{1}{3}}(\alpha R)^{\frac{1}{3}} \tag{9.7}$$

yields the most critical value of c, that which corresponds to the *least* damped solution. The results, calculated by means of (9.7) are compared to corresponding values from a differential analyser (Corcos 1952) in figure 3. Unfortunately, it was not possible to operate the analyser above $\alpha R = 1250$, and thus $(\alpha R)^{\frac{1}{3}}$ cannot be said to be extremely large for this case. Nevertheless, the results agree surprisingly well. The small amount of scatter in the computer data should be charged to the computer itself.

The close concurrence between the results of the differential analyser and the analytical formula (9.7) derived above give considerable confidence in the accuracy of the asymptotic methods. At low Reynolds numbers, the problem lends itself especially well to study with a differential analyser. It was noted in the course of the computer study that there were few eigenvalues corresponding to a given value of α and αR, and that the solutions yielded large decays in time. It was also clear from the character of the solutions of the differential equations that none would meet the boundary conditions with a negative value of c_i (corresponding to an unstable eigenvalue).

Thus it must be concluded that Poiseuille flow damps infinitesimal axially symmetric disturbances of the type assumed here.

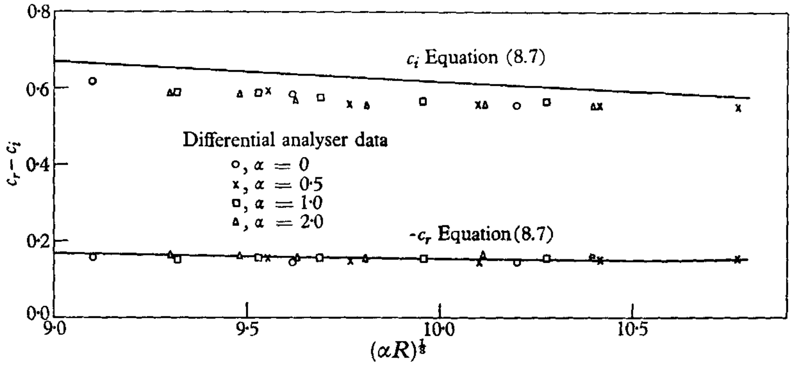

FIGURE 3. c_r, c_i *vs* $(\alpha R)^{\frac{1}{2}}$.

10. Conclusions

The purpose of a stability analysis is to provide information about the phenomenon of transition from a laminar flow to a turbulent one. In recent years it has been more fully realized that while the small-disturbance theory has a legitimate basis in fact (Schubauer & Skramstadt 1947; Liepmann 1943; Laufer 1956), it plays only a fragmentary role in the description of transition.

To begin with, if the oscillations grow, they eventually reach an amplitude beyond which linearization of the equations of motion is inadmissible. At this point we lose even qualitative information about the further development of the disturbances. They may grow, settle at some finite amplitude or decay. They may appear as localized turbulent spots or spearheads (Emmons 1951; Schubauer & Klebanoff 1955; Rotta 1956), and these may or may not spread. Here an entirely different type of analysis seems to be required to describe the spatial growth of a finite, even fully developed disturbance rather than the temporal growth of an infinitesimal one. Finally, even when small disturbances are guaranteed not to grow within a region of flow, disturbances of finite (but not necessarily large) amplitude may be 'imported'. The free stream turbulence of a wind tunnel may induce in boundary layers oscillations large enough to invalidate the conclusions of a small-disturbance analysis. In the problem which occupies us, finite disturbances may very well propagate downstream from the entrance of the pipe. There are good grounds (Tatsumi 1952) for believing that, for some Reynolds numbers, as the boundary layer grows on the walls of the pipe entrance, it is capable of amplifying infinitesimal disturbances before they reach the downstream portion of the pipe where fully developed parabolic flow prevails. Are we able to deduce from the analysis presented here that contamination from upstream is indeed the only means by which disturbances in a pipe can acquire a finite amplitude? Unfortunately, this analysis taken by itself does not justify such strong conclusions. On the other hand, a recent set of experiments performed by Leite (1959) on fully developed pipe flow usefully complements the analysis.

There are three main reasons why one might hold that the small-disturbance analysis of fully developed pipe flow is incomplete.

(*a*) Axially symmetric disturbances have been assumed. This is an arbitrary albeit necessary restriction designed to permit separation of variables. No analytic considerations have been advanced to bolster the hope that asymmetric disturbances behave as symmetric ones do. However, Leite (1959) purposely introduced small disturbances with no symmetry. They became more axially symmetric as they progressed downstream. Thus the asymmetric part of the disturbances decayed faster than the symmetric part.

(*b*) It has been pointed out that the analysis yields a finite set of eigenvalues and hence of eigenfunctions for any given value of α and of αR. This means that arbitrary initial disturbances which are held to satisfying continuity cannot be represented as we have assumed by a sum of elemental solutions of the form

$$\psi(r,x,t) = \phi(r)\,e^{i\alpha x}.$$

For instance, $$\psi_{\text{initial}} = f(r,R)\cos\alpha_1 x,$$

where $f(r,R)$ is an arbitrary continuous function of r subject to our boundary conditions, cannot be represented in the analysis since, according to our results, $f(r)$ can only be the linear combination of a finite number of eigenfunctions, i.e.

$$f(r) = \sum_1^n A_n\,\phi_n(\alpha_1, R, r).$$

Leite found that the rate of decay for the disturbances he introduced was given with relatively good accuracy by the present analysis. This would seem to indicate that the rates of decay are not very sensitive to the form of the disturbance so long as its amplitude is small.

(*c*) Finally, in the analysis disturbances are assumed periodic for

$$-\infty < x < \infty.$$

This is only one class of perturbations. For instance, disturbances introduced by a small roughness on the wall or artificially as in Leite's pipe occur, instead, at a given downstream section. If they are heavily damped or heavily amplified, say if the amplitude varies appreciably in one wavelength, the distinction may be important. This criticism naturally applies also to the study of boundary layer stability. One should perhaps not expect as good agreement between theory and experiment when the decay or amplification rates are large as say for the determination of a neutral curve. For pipe flows, since the experiments suggest heavy damping, it might be instructive to analyse the problem anew with disturbances of the form

$$\psi(r,x,t) = \phi(r)\,e^{-\nu x + i\omega t},$$

where ν is complex and ω is real.

Leite's experiments and the present work taken together do not amount to a proof that small disturbances cannot grow in a fully developed pipe flow, but they provide a very strong indication that this is the case.

Appendix

The Stokes phenomenon

Since, in general, the point $W = c$ corresponds to a complex value of r, we shall extend r into the complex domain by analytic continuation, and we shall make use of the fact that a true (i.e. exact) solution of (2.1) is single-valued analytic and continuous at and around the 'Stokes point' $W = c$. This avoids the indeterminacy introduced with the asymptotic solutions. The following development which is related to some of Langer's work is essentially due to Furry (1947), and the general approach is known as the W.K.B. method.

Let $W = c$ when $r = k$ and define

$$U = \int_k^r \sqrt{\{i(W-c)\}}\,dr.$$

Then if $(r-k) = \beta$, we have in the close neighbourhood of k

$$U = \int_0^{r-k} \sqrt{\{-i(2k)\beta\}}\,d\beta = \tfrac{2}{3}\sqrt{(-2ik)}\,\beta^{\frac{3}{2}}.$$

We seek the lines in the complex plane for which U is a pure real number. If we define $\arg(i) = \frac{1}{2}\pi$, the approximate expression for U above shows that at least in the vicinity of k these lines are such that

$$\arg\beta = -\tfrac{1}{2}\pi + \tfrac{2}{3}N\pi + \arg\sqrt{k}.$$

That is, there are three such lines which make 120° angles with each other near the Stokes point (figure 1). Away from the Stokes point these lines will be curved, because the approximation for U above will not hold for β-large. The sign of U alternates from line to line, so that for a given line it depends on the number of revolutions around the point k. These lines will be designated as the Stokes lines. In a similar way, U is purely imaginary near the Stokes point on lines such that $\arg(\beta) = \frac{1}{2}\pi + \frac{2}{3}N\pi + \arg\sqrt{k}$. As one proceeds out along a Stokes line the exponential terms $\exp[\sqrt{(\alpha R)}\,U]$, $\exp[-\sqrt{(\alpha R)}\,U]$ in the asymptotic expressions become either very large or very small depending on the sign of U associated with that line. This a consequence of the assumption that αR is a large number. Thus, if we were to traverse the β-plane in such a manner as to cross Stokes lines, we would successively see the coefficient of A in (4.3) become much larger than the coefficient of B, then the coefficient of B overpower the coefficient of A, etc. When one is on a Stokes line, the small exponential term is in fact smaller than the order of approximation of the asymptotic representation. On the other hand, on the lines of pure imaginary U, the coefficients of A and B have precisely the same magnitude. It is apparent from this observation that if we need to change our coefficients A and B within the interval, the proper place to modify them will be on the Stokes lines. There the coefficient of the small term could be modified without materially altering the value of the sum of the two asymptotic expressions, and thus without introducing a discontinuity in the representation.

We now inquire whether the original ratio A/B need be modified within the interval $(0 < r < 1)$. We define the Stokes lines and the lines of pure imaginary

U as shown in figure 1. Further we choose $U > 0$ on S_1, hence $U < 0$ on S_2, $U > 0$ on S_3, for the first revolution; and $U < 0$ on S_1' after one complete revolution from S_1. If the asymptotic solution (4.3) is represented in Region I by

$$\phi_a = \frac{r^{\frac{1}{2}}}{(W-c)^{\frac{5}{4}}}[A_1\chi_+(k,r)+B_1\chi_-(k,r)],$$

it can be represented in any Stokes region by a linear combination of the two terms. Thus

$$A_{\mathrm{II}} = aA_1+bB_1, \quad B_{\mathrm{II}} = cA_1+dB_1,$$

where a, b, c, d are constants. However, the continuity of our representation requires that the coefficient of the small term only be modified on the corresponding Stokes lines. Therefore, according to our convention we have

$$\begin{aligned} A_{\mathrm{II}} &= A_1, \\ B_{\mathrm{II}} &= B_1+\alpha A_{\mathrm{I}}, \\ A_{\mathrm{III}} &= A_{\mathrm{II}}+\mu B_{\mathrm{II}} = A_1+[B_1+\alpha A_1]\mu, \\ B_{\mathrm{III}} &= B_{\mathrm{II}} = B_{\mathrm{I}}+\alpha A_{\mathrm{I}}, \end{aligned}$$

and at the initial Stokes line after a full revolution counterclockwise

$$\begin{aligned} A_{\mathrm{I}}' &= A_{\mathrm{III}} = A_{\mathrm{I}}+\mu[B_1+\alpha A_1], \\ B_{\mathrm{I}}' &= B_{\mathrm{III}}+\gamma A_{\mathrm{III}} = B_1+\alpha A_{\mathrm{I}}+\gamma\{A_1+\mu[B_1+\alpha A_1]\}. \end{aligned}$$

On the other hand, after one revolution A has become the coefficient of the small term, B that of the large term, and the argument of the expression $(W-c)^{\frac{5}{4}}$ has gained an angle $5\pi/2$. For the solutions on A_{I} and A_{I}' to be the same requires that

$$\begin{aligned} A_{\mathrm{I}}' &= iB_1, \\ B_{\mathrm{I}}' &= iA_1. \end{aligned}$$

All the above relations will be compatible only if

$$\alpha = \mu = \gamma = i. \tag{A 1}$$

The present arguments show that for the representation to be continuous, one of the constants A or B must be modified every time a Stokes line is crossed. This property is known as the Stokes phenomenon. We now have a series of connexion formulae which permit us to evaluate this discontinuity in A and B on the Stokes lines. Which connexion formula will be used depends on the Stokes regions within which the end points $r = 0$ and $r = 1$ fall, and thus on the eigenvalue yet to be found. Once an arbitrary decision is made, it can be checked *a posteriori*; i.e. it is a correct choice only if it yields eigenvalues consistent with the assumptions. There are obviously nine cases to be considered, but it is now a simple matter to determine which is acceptable. One is guided partly by the expectation that the phase velocity of the disturbance (c_r) is positive and less than the maximum velocity of the main flow: $0 < c_r < 1$. This suggests that $r = 0$ is either in Region II or III. A systematic investigation by the authors indicated that eigenvalues existed only if they were such that $r = 0$ belonged to Region III and $r = 1$ to Region I. For this case our connexion formulae yield

$$A_1 = iB_{\mathrm{III}}+A_{\mathrm{III}}, \quad B_1 = -iA_{\mathrm{III}}. \tag{A 2}$$

If A_{III} and B_{III} are known, A_{I} and B_{I} are given by (A 2) and hold from the neighbourhood of the singularity to the wall. According to equations (4.9) and (4.3), reversing the order of integration, and changing the lower limit of integration, we find that

$$\begin{aligned} A_1 &= C_2[\exp(\tfrac{3}{4}\pi i)\,\chi_-(k,0) - i\exp(-\tfrac{3}{4}\pi i)\,\chi_+(k,0)], \\ B_1 &= C_2[-i\exp(\tfrac{3}{4}\pi i)\,\chi_-(k,0)], \end{aligned} \tag{A 3}$$

where C_2 is a constant.

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